1 Introductory linear algebra: problem set

Question 1

Write the vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ as a linear combination of the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

Question 2

Show that any set of vectors which includes $\mathbf{0}$ must be linearly dependent.

Question 3

Given the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 10 & -3 \\ 10 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 8 \end{pmatrix} \quad E = \begin{pmatrix} 3 & -1 & 9 \\ 6 & -1 & 7 \\ 3 & 10 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 3 & 9 \\ 1 & -1 \\ 10 & 1 \end{pmatrix}$$

and the vectors

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Find the following matrix quantities, or explain why they do not exist

$$\begin{array}{llllllll} AB, & AC, & AD, & DA, & A+B, & 2A-5B, & A+D, & BC \\ DF, & FD, & AE, & A\mathbf{x}, & \mathbf{x}B, & D\mathbf{y}, & E\mathbf{y}, & F\mathbf{y}, & B+C. \end{array}$$

(b) Verify (by calculating the left-hand and right-hand sides of both expressions and checking that they are equal)

$$\begin{aligned} (AB)C &= A(BC) \\ A(B+C) &= AB+AC \end{aligned}$$

Question 4

(a) If $A^{-1}(X+B)C^{-1} = I$ and both A and C are invertible, solve for X in terms of A, B and C.

Question 5

Consider the linear system

$$\begin{aligned} x + 2y - 3z &= -1 \\ 2x + 2y + 9z &= 5 \\ 5x + 6y - z &= 1 \end{aligned}$$

(a) Write the system in matrix-vector form $A\mathbf{v} = \mathbf{b}$, where A and $\mathbf{b}$ are to be determined, and where the unknown vector $\mathbf{v}$ has elements x, y, z.

(b) By calculating the determinant of A, establish whether a solution exists for this system.

Question 6

For each of the following linear systems, use Gaussian elimination to determine whether it is consistent, and if so evaluate its solution.

(a)

$$\begin{aligned} 2y - 8z &= 16 \\ 2x - 3y + 2z &= 1 \\ 5x - 8y + 7z &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= 3 \\ 3x_1 + x_2 - x_4 &= 1 \\ -3x_2 + 2x_3 - 2x_4 &= -7 \\ -4x_1 - x_2 + 2x_4 &= 3 \end{aligned}$$

Question 7

Consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 7 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 7 & 4 \\ 6 & -1 & 4 \\ 3 & 1 & -3 \end{pmatrix}.$$

(a) Find the determinants of the matrices A and B.

(b) Calculate the matrices AB and BA.

(c) Verify, for these matrices, the formula

$$\det(AB) = \det(BA) = \det(A)\det(B).$$

(Note that this is actually the case for all identically-sized square matrices A and B).

(d) Verify that $\det(A^T) = \det(A)$, and $\det(B^T) = \det(B)$. (In fact it is true for any matrix M that $\det(M^T) = \det(M)$. Thinking about how the determinant is calculated, can you see why?)

Question 8

(This question leads to a geometric interpretation of the determinant in 2D)

Consider how the set of points defining the vertices of the unit square $\{(0,0),(1,0),(0,1),(1,1)\}$ are affected by pre-multiplication by the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. What is the area of the resulting shape?

Question 9

Consider the 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. Suppose that $B = A^{-1}$, so that

$$AB = I$$

By writing out this matrix multiplication in full, show that it reduces to two pairs of simultaneous equations. Show that for these equations to have a solution (in other words, for A to be invertible) we must have $\det(A) \neq 0$, in which case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Question 10

Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$

Question 11

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

Question 12

Let A be an invertible matrix.

(a) If λ is an eigenvalue of A, show that λ^{-1} is an eigenvalue of A^{-1}.

(b) Show that A^T is invertible with inverse $(A^T)^{-1} = (A^{-1})^T$.

Question 13

Consider the matrix

$$A = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 8 & 2 \\ 0 & -3 & 1 \end{pmatrix}.$$

Find the eigenvalues of A. Demonstrate that the matrix satisfies the equation

$$A^3 - 12A^2 + 41A - 42I \quad = \quad 0,$$

where I is the 3×3 identity matrix and where 0 is the 3×3 zero matrix. What do you notice?

Question 14

Consider the following vectors:

$$\mathbf{x} \quad = \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 5 \\ 0 \end{pmatrix},$$

(a) Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be the eigenvectors of the matrix $A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$
Find constants α, β, γ and δ such that $\mathbf{x} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$ and $\mathbf{y} = \gamma\mathbf{v_1} + \delta\mathbf{v_2}$.

(b) Find the vector $\mathbf{z} = A^5\mathbf{x}$.

(c) What is the behaviour of $A^m\mathbf{y}$ as $m \to \infty$?

Question 15

Let $B = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, and $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

(a) If $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ are the eigenvectors of B (which you found in Question 1), find constants α, β, γ such that $\mathbf{x} = \alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3}$.

(b) Find the vector $\mathbf{z} = B^{10}\mathbf{x}$. Comment on its direction.