Mathematical Biology Linear Algebra Select Solutions

Georgeos Hardo

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Question 6

Part (a)

Use Gaussian elimination to determine whether this system is consistent:

$$\begin{aligned} 2y + 8z &= 16 \\ 2x - 3y + 2z &= 1 \\ 5x - 8y + 7z &= 1 \end{aligned}$$

Rewrite this in augmented form:

$$\left[\begin{array}{ccc|c} 0 & 2 & -8 & 16 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right]$$

$$R_1 \iff R_3$$

$$\left[\begin{array}{ccc|c} 5 & -8 & 7 & 1 \\ 2 & -3 & 2 & 1 \\ 0 & 2 & -8 & 16 \end{array}\right]$$

$$R_2 \implies -5R_2 + 2R_1$$

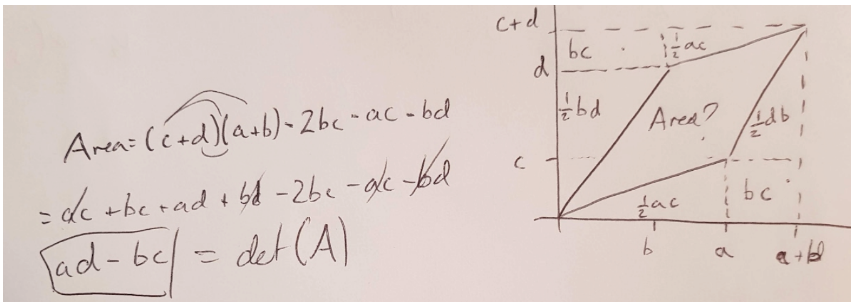
$$\left[\begin{array}{ccc|c} 5 & -8 & 7 & 1 \\ 0 & -1 & 4 & -3 \\ 0 & 2 & -8 & 16 \end{array}\right]$$

$$R_3 \implies R_3 + 2R_2$$

$$\left[\begin{array}{ccc|c} 5 & -8 & 7 & 1 \\ 0 & -1 & 4 & -3 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

Question 8

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{bmatrix}$$



Question 9

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, $AB = I_3$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And so we have pair of systems of 2 equations:

$$ax + bz = 1$$
$$cx + dz = 0$$

and

$$ay + bt = 1$$
$$cy + dt = 0$$

Solve for (x, y, z, t), and substitute back into the original matrix to find:

$$B = \begin{pmatrix} \frac{d}{\det(A)} & \frac{-b}{\det(A)} \\ \frac{-c}{\det(A)} & \frac{a}{\det(A)} \end{pmatrix}$$

Factoring out $\det(A)$ we see that indeed $B = A^{-1}$

$$B = A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $\det(A) = 0$, meaning $ad = bc$, then the above equations will have no or infinite solutions. Moreover division by zero is disallowed.

Question 10

Part (a)

We want to solve the equation $A\mathbf{v} = \lambda\mathbf{v} \implies (A - I_2\lambda)\mathbf{v} = 0$

We are given

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$$

And therefore

$$(A - I_2\lambda) = \begin{pmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{pmatrix}$$

We are interested in finding the Eigenspace of this matrix, corresponding to an infinite set of solutions. This means we would like to solve the equation:

$$\det(A - I_2\lambda) = 0 = \begin{vmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{vmatrix}$$

This yields the characteristic polynomial of the matrix:

$$\lambda^2 - 7\lambda + 6 = 0$$

Solving this polynomial gives the Eigenvalues of the matrix: $\lambda_1 = 6$ and $\lambda_2 = 1$

Let us now find the Eigenvectors for λ_1. We can do this by substituting in λ_1 and solving $(A - I_2\lambda)\mathbf{v} = 0$

$$\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

This yields a set of simultaneous equations:

$$-3x + 6y = 0$$
$$x - 2y = 0$$

Which can be reduced to

$$x - 2y = 0$$
$$0 = 0$$

Essentially the system can be described by 1 equation, meaning that one variable is free (can be anything). In this case let us choose a variable $y = \alpha$. Therefore

$$x = 2\alpha$$

We can write our basis eigenvector vector as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{v}_1$$

This can be repeated for the second eigenvalue to show that the second eigenvector is:

$$\mathbf{v}_2 = \alpha \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Part (b)

By the same process as before, we want to solve:

$$\det(B - I_3\lambda) = 0 = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - 0 + 2\begin{vmatrix} 0 & 1-\lambda \\ 2 & 0 \end{vmatrix}$$

Solving this characteristic polynomial:

$$\det(B - I_3\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6 = (\lambda - 1)(\lambda - 3)(\lambda + 2)$$

And so the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$

Substituting λ_1 into $(B - I_3\lambda)\mathbf{v} = 0$:

We can collapse the system of equations down to:

$$\begin{aligned} -x + 2z &= 0 \\ -2y &= 0 \\ 0z &= 0 \end{aligned}$$

Note that z can take any value, but y can only be 0. Therefore let us set $\alpha = z$. Therefore:

$$2\alpha = x$$

And our eigenvector is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\alpha \\ 0 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \mathbf{v}_1$$

Substituting λ_2 into $(B - I_3\lambda)\mathbf{v} = 0$ the system is:

$$\begin{aligned} 4x + 2z &= 0 \\ 3y &= 0 \\ y &= 0 \end{aligned}$$

The only consistent value is that $y = 0$. Therefore let us once again set $\alpha = z$ and note that:

$$x = -\frac{1}{2}\alpha$$

Therefore our eigenvector is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\alpha \\ 0 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \mathbf{v}_2$$

For the final eigenvalue, the system of equations is simply:

$$\begin{aligned} x + 2z &= 0 \\ 2x - 2z &= 0 \\ 0y &= 0 \end{aligned}$$

The only consistent solutions are that x and z both equal 0. The only free variable is y. Therefore:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{v}_3$$

Question 11

Let us find the eigenvalues and eigenvectors of:

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

With the same methods as above, the characteristic polynomial is:

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = (\lambda + 1)(\lambda + 1)(\lambda - 8) = 0$$

The eigenvalues are therefore $\lambda_1 = 8$ and $\lambda_2 = \lambda_3 = -1$. We have a repeated root, and so one of our eigenspaces will encompass an entire 2D plane, and not just a single 1D line.

For λ_1 we have the following system of equations:

$$\begin{aligned} -5x + 2y + 4z &= 0 \\ -18x + 18z &= 0 \\ 0 &= 0 \end{aligned}$$

Let us set $z = \alpha$. Therefore it is clear that $x = \alpha$ and $y = \alpha/2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ \frac{1}{2}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \mathbf{v}_1$$

We have a slightly different situation for the second eigenvalue, in this case the system reduces to:

$$\begin{aligned} 4x + 2y + 4z &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

And so we have two free variables. Let us choose $\alpha = y$ and $\beta = z$. Therefore:

$$x = -\frac{1}{2}\alpha - \beta$$

Substitution into the basis vector:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{v}_2$$

Question 12

Part (a)

We are asked to show that if λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}. An eigenvalue satisfies the equation:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Simply dividing through by *lambda* and inverting A gives:

$$\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$$

Part (b)

We are asked to show that $(A^T)^{-1} = (A^{-1})^T$.

Recall that $(AA^{-1})^T = I^T$. Then $(A^{-1})^T A^T = I$ (from $(AB)^T = B^T A^T$). Therefore it follows that $(A^T)^{-1} = (A^{-1})^T$.

Question 14

Part (a)

From a previous question, we know that

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 6$, and $\lambda_2 = 1$, and eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

To find the constants α, β, γ, δ, such that $\mathbf{x} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ and $\mathbf{y} = \gamma\mathbf{v}_1 + \delta\mathbf{v}_2$, we need to solve a system of equations. For $\mathbf{x}$:

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 1 \end{pmatrix} \implies \begin{aligned} 2\alpha - 3\beta &= 1 \\ \alpha + \beta &= 2 \end{aligned}$$

hence $\alpha = \frac{7}{5}$, $\beta = \frac{3}{5}$. Similarly for $\mathbf{y}$:

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \delta \begin{pmatrix} -3 \\ 1 \end{pmatrix} \implies \begin{aligned} 2\gamma - 3\delta &= 5 \\ \gamma + \delta &= 0 \end{aligned}$$

hence $\gamma = 1$, $\delta = -1$.

Part (b)

Recall the derivation from the lectures:

$$\mathbf{z}_m = \alpha_1 \lambda_1^m \mathbf{v}_1 + \alpha_2 \lambda_2^m \mathbf{v}_2 + \cdots + \alpha_n \lambda_n^m \mathbf{v}_n$$

where $\mathbf{z}_m = A^m \mathbf{z}$

We are asked to find $\mathbf{z} = A^5 \mathbf{x}$. Therefore using the previous result:

$$\begin{aligned}\mathbf{z} &= \frac{7}{5} \cdot 6^5 \cdot \mathbf{v}_1 + \frac{3}{5} \cdot 1^5 \cdot \mathbf{v}_2 \\ &= \frac{7}{5} \cdot 7776 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{5} \cdot 1 \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 21771 \\ 10887 \end{pmatrix}\end{aligned}$$

Part (c)

We are asked about the behaviour of $A^n \mathbf{y}$ as $n \to \infty$. From the previous result we know that:

$$A^n \mathbf{y} = 6^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 1^n \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

As n gets very large, $6^n \gg 1^n$, and so the second term in $A^n \mathbf{y}$ becomes vanishingly small. Therefore for large n:

$$A^n \mathbf{y} \approx \begin{pmatrix} 6^n \cdot 2 \\ 6^n \end{pmatrix}$$

, which points in the direction of $\mathbf{v}_1$.

Question 15

We are given: $B = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Part (a)

We are asked to find the parameters such that $\mathbf{x} = \alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3$. We know the eigenvectors from a previous question and can plug them in such that:

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} &= \\ &= \alpha \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}\end{aligned}$$

Using Gaussian elimination, we can determine that the $\alpha = \frac{5}{2}$, $\beta = \frac{3}{5}$, $\gamma = 2$

Part (b)

We would like to $\mathbf{z} = B^{10} \mathbf{x}$. From a previous question we determined the eigenvalues were $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = 1$. From our previous result we can write:

$$\begin{aligned}\mathbf{z} &= \frac{1}{5} \cdot 3^{10} \cdot \mathbf{v}_1 + \frac{3}{5} \cdot (-2)^{10} \cdot \mathbf{v}_2 + 2 \cdot 1^{10} \cdot \mathbf{v}_3 \\ &= \frac{1}{5} \cdot 59049 \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{5} \cdot 1024 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 2 \cdot 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 24234 \\ 2 \\ 10581 \end{pmatrix}\end{aligned}$$

The vector points very largely in the direction of $\mathbf{v}_1$. Therefore the magnitude of the largest eigenvalue determines the main relative direction of the vector.