Sheet A, Q2. Exponential growth

Question 2. Exponential growth

A model used to represent the dynamics of a population of size $y(t)$ at time t is

$$\frac{dy}{dt} = \beta y,$$

where β is a positive parameter.

(a) Briefly explain the key biological assumption underlying this model.

(b) Solve the model, assuming that the initial population size is y_0.

(c) Sketch the solution of the model, illustrating carefully any effect(s) of changes to β and y_0.

(d) Find an expression for the time taken for the population to increase by a factor of 10.

a) No dependence of growth rate on any (density) interaction, and therefore the relative growth rate: $\frac{1}{y}\frac{dy}{dt}$ is constant.

b) Separate variable: $\frac{dy}{dt} = \beta y \Rightarrow \frac{dy}{y} = \beta dt$

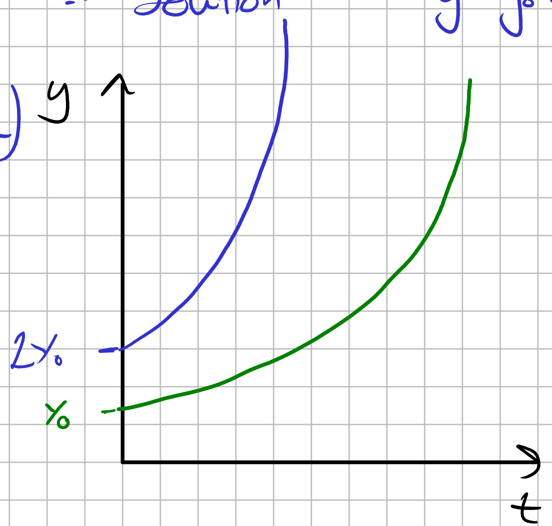
Integrate $\int \frac{dy}{y} = \int \beta dt \Rightarrow \ln(y) = \beta t + c$

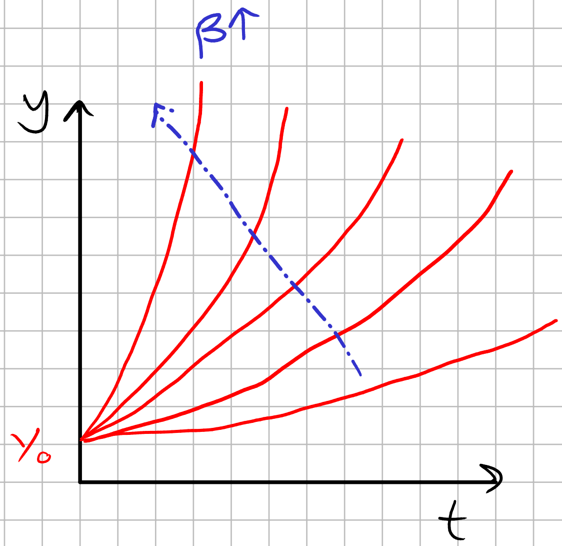
Exponentiate $y = e^{\beta t + c} = e^c e^{\beta t} = Ce^{\beta t}$

Set $y = y_0$ @ $t = 0$ $\quad y_0 = Ce^0 = C$

$\therefore$ Solution: $y = y_0 e^{\beta t}$

c)





d) Expression for $y_0 \to 10y_0$ time (t_{10})

$$10\cancel{y_0} = \cancel{y_0}e^{\beta t_{10}} \Rightarrow 10 = e^{\beta t_{10}}$$

$$\ln(10) = \beta t_{10} \Rightarrow t_{10} = \frac{\ln(10)}{\beta}$$ (Similar to the doubling time!)

Question 3. Radiocarbon dating

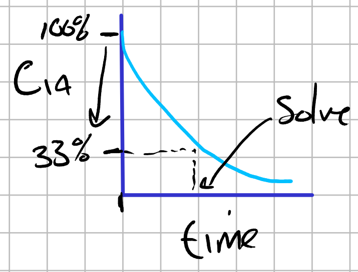
Write and solve a differential equation describing the decay of ^{14}C. A unit volume sample of an old piece of oak taken from a museum has an activity of ^{14}C that is one third that of a similar volume of a modern piece of oak. Given that the half-life for ^{14}C is 5730 years, how old is the museum piece?

$$\frac{dy}{dt} = -\beta y \Rightarrow \int \frac{dy}{y} = -\int \beta t \Rightarrow \ln(y) = -\beta t + c$$

$$y = ce^{-\beta t}, \Rightarrow y = y_0 e^{-\beta t}$$ (as before)

We want to know age, ∴ solve for t

Activity of piece is $1/3$ that of a modern piece



$$\underset{\text{aged}}{\frac{1}{3}} = \underset{\text{current}}{1} \cdot e^{-\beta t}$$? Need β.

Half life = 5730 yrs, Recall $t_{1/2} = \frac{\ln(2)}{\beta}$ (as before)

$$\therefore \beta = \frac{\ln(2)}{5730\,\text{yr}} \approx 9.08 \times 10^{-3}\ \text{yr}^{-1}$$

Solve for t $\ln\left(\frac{1}{3}\right) = -\beta t \Rightarrow \frac{\ln(1/3)}{-\beta} = t \Rightarrow t \approx 9000$ years

Question 4. Immigration and birth or death

A group of organisms living in an isolated habitat is subject to immigration at constant rate α, reproduction at per capita rate β, and death at per capita rate γ, where α, β and γ are positive constants

(a) Write down an expression describing the evolution of $Y(t)$, the size of the population at time t.

(b) Solve your model to find $Y(t)$, given that the habitat was initially empty.

(c) Sketch your solution and biologically interpret the changes to your sketches as the values of the parameters are changed.

(d) What happens if $\beta = \gamma$?

Rate of change = in − out

in = immigration + reproduction

out = death

α is a constant rate, units organisms/time

β, γ are per capita rates (organisms/time)/organism

$\therefore$ Let organisms be represented by y

$$\frac{dy}{dt} = \alpha + \beta y - \gamma y = \alpha + y(\beta - \gamma)$$

$\left[\text{Recall } \int \frac{1}{ax+b}\,dx = \frac{\ln(ax+b)}{a} \right.$

Solving: $\displaystyle\int \frac{dy}{\alpha + y(\beta-\gamma)} = \int dt$

$$\Rightarrow \frac{\ln(\alpha + y(\beta-\gamma))}{(\beta-\gamma)} = t + c, \quad \text{Let } \beta - \gamma = \delta$$

$(\delta c = c)$

$$\ln(\alpha + \delta y) = \delta t + c \Rightarrow \alpha + \delta y = e^{\delta t} \cdot c$$

@ $t = 0$, $y = y_0$ $\therefore$ $\alpha + \delta y_0 = c$, we are told $y_0 = 0$

$\therefore c = \alpha$

(NOTE: depending when you set the initial conditions you may get a different value for c if you substituted before exponentiating)

$$\alpha + \delta y = \alpha e^{\delta t} \Rightarrow y = \frac{\alpha(1-e^{\delta t})}{\delta}$$

Note, the worked solution from the lecturer gives:

$y = \frac{\alpha}{\omega}(1-e^{-\omega t})$, which is the same because

$$\omega = -\delta \qquad [\beta - \gamma = -(\gamma - \beta)]$$

Both methods give same solution.

c)

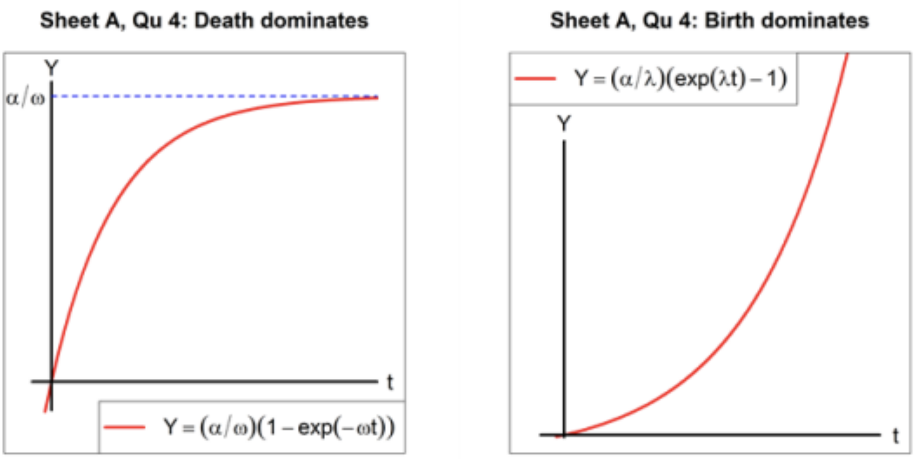


Figure 4. Sketches for Sheet A, Qu 4. The form of the graph depends on whether $\omega \lessgtr 0$, i.e. whether death or birth dominates within the habitat.

(c) Note that ω is the net rate of per capita death (which can be positive or negative depending on the relative sizes of γ and β). The form of the sketch depends on whether $\omega \lessgtr 0$ (see Figure 4).

At $\beta = \gamma$, $\frac{dy}{dt} = \alpha \Rightarrow y = \alpha t + c$, $y = 0$ @ $t = 0$

$\therefore c = 0$

$\boxed{y = \alpha t}$

Population increase from immigration only.

Question 5. Newton's law of cooling

A body is found indoors at 16.00h. The temperature of the body (T) is 27^{o}C; by 19.00h, it has fallen to 24^{o}C. The room temperature is assumed to have remained constant at 10^{o}C and the original body temperature may be assumed to be 37^{o}C.

(a) Assuming that the rate of cooling of the body was proportional to the difference between room temperature and body temperature, construct a simple equation for $T(t)$ and solve it.

(b) Use the information about cooling between 16.00h and 19.00h to estimate the rate parameter for the cooling process and then estimate the time of death to the nearest hour.

a) Rate of cooling $\propto \Delta T$, where $\Delta T = T_{ROOM} - T_{BODY}$

$= T_R - T_B$

$\therefore \frac{dT_B}{dt} \propto T_R - T_B \Rightarrow \frac{dT_B}{dt} = k(T_R - T_B) = kT_R - kT_B$

T_B is a function of t, T_R is constant (10°C)

$$\int \frac{dT_B}{kT_R - kT_B} = \int dt \Rightarrow \frac{-\ln(kT_R - kT_B)}{k} = t + C$$

$\therefore kT_R - kT_B = e^{-kt} \cdot C$

Let 16:00 be $t=0$, $T_{B0} = 27°C$, $T_R = 10°C$ ALWAYS

$\therefore k \cdot 10 - k \cdot 27 = k(-17) = c$

Full model $\therefore$ $10\cancel{k} - \cancel{k}T_B = -17\cancel{k}e^{-kt} \Rightarrow T_B = 17e^{-kt} + 10$

We need k now. In 3 hours, body cools by 3°C

$t=0$ is 16:00 $\therefore$ 19:00 is $t=3$

$24 = 17e^{-k \cdot 3} + 10 \Rightarrow \frac{14}{17} = e^{-k \cdot 3} \Rightarrow \ln\left(\frac{14}{17}\right) = -3k$

$k = 0.0647 h^{-1}$

Now, when was the time of death?

Assume body was 37°C when it died (t_d)

Plug into model:

$$37 = 17e^{-0.0647 t_d} + 10$$

$$\Rightarrow \frac{27}{17} = e^{-0.0647 t_d} \Rightarrow \frac{\ln(27/17)}{-0.0647} = t_d$$

$t_d = -7.15$ hours

We defined 16:00 to be $t=0$

$\therefore$ $t = -7.15$ is approximately 09:00