Mathematical Biology Worksheet 1 Solutions (Part I)

Georgeos Hardo

October 2020

Question 1

(a)

$$P(\text{coyote first}) = \frac{30}{100} = \frac{3}{10}$$

(b)

$$P(\text{Wolf first}) = \frac{10}{100} = \frac{1}{10}$$

(c) Question weirdly worded. Acceptable solutions:

$$P(\text{Fox second}|(\text{Wolf} \cup \text{Coyote})\ \text{first}) = \frac{60}{99} = \frac{20}{33}$$

Or

$$P(\text{Fox second} \cap (\text{Wolf} \cup \text{Coyote})\ \text{first}) = \left(\frac{10}{100} + \frac{30}{100}\right) \cdot \frac{60}{99} = \frac{8}{33}$$

Question 2

(a) Using the Law of Total Probability, and the fact that events are independent:

$$\begin{aligned} P(\mathrm{R}_1 \to 0) &= P(\mathrm{R}_1 \to 0 \cap S = 0) + P(\mathrm{R}_1 \to 0 \cap S = 1) \\ &= 0.98 \cdot \frac{1}{3} + 0.02 \cdot \frac{2}{3} \\ &= 0.34 \end{aligned}$$

Tree diagram also acceptable!

(b)

(c) Using the Law of Total Probability, and the fact that events are independent ($P(A \cap B) = P(A)P(B)$):

$$\begin{aligned} P(\mathrm{R}_2 \to 1) &= P(\mathrm{R}_2 \to 1 \cap S = 0) + P(\mathrm{R}_2 \to 1 \cap S = 1) \\ &= 0.03 \cdot \frac{1}{3} + 0.97 \cdot \frac{2}{3} \\ &= 0.657 \end{aligned}$$

Tree diagram also acceptable!

(d) We want to compute $P(S = 0|\mathrm{R}_1 \to 0)$ and $P(S = 1|\mathrm{R}_2 \to 1)$ (The probabilities that the signal is the same as the one being reported). We can use Bayes Theorem for this:

$$P(S = 0|\mathrm{R}_1 \to 0) = \frac{P(\mathrm{R}_1 \to 0|S = 0) \cdot P(S = 0)}{P(\mathrm{R}_1 \to 0)}$$

We can find $P(\mathrm{R}_1 \to 0|S = 0)$ because the if the signal is 0, then R_1 has a 0.98 chance of being correct, and so $P(\mathrm{R}_1 \to 0|S = 0) = 0.98$.

From part (a) we know $P(\mathrm{R}_1 = 0.34$, and from the question we know $P(S = 0) = \frac{1}{3}$. Thus:

$$P(S = 0|\mathrm{R}_1 \to 0) = \frac{0.98 \cdot \frac{1}{3}}{0.34} = 0.96$$

We can calculate $P(S = 1|\mathrm{R}_2 \to 1)$ similarly:

$$P(S = 1|\mathrm{R}_2 \to 1) = \frac{P(\mathrm{R}_2 \to 1|S = 1) \cdot P(S = 1)}{P(\mathrm{R}_2 \to 1)} = \frac{0.97 \cdot \frac{2}{3}}{0.657} = 0.984$$

Question 3

(a) There are 12 healthy trees and 10 unhealthy trees. Therefore the ways to select 3 healthy trees AND 2 unhealthy trees is:

$$\binom{12}{3} \cdot \binom{10}{2} = 9900$$

(Use the nCr button on your calculator.)

(b) The probability of choosing the previous condition of 5 saplings is simply 9900 divided by the total number of ways to choose 5 saplings. Remember, order doesn't matter so this is a **combination**

$$\frac{\binom{12}{3} \cdot \binom{10}{2}}{\binom{22}{5}} = \frac{9900}{26334} = 0.376$$

(c) We want to choose 5 healthy trees AND 0 unhealthy trees:

$$\binom{12}{5} \cdot \binom{10}{0} = 792$$

(d) Similarly to question (b), the probability of this occurring is:

$$\frac{\binom{12}{5} \cdot \binom{10}{0}}{\binom{22}{5}} = \frac{792}{26334} = 0.03$$

Question 4

We want $P(\text{Spam}|\text{CTO})$. Here are the things we know:

- $P(\text{Spam}) = 0.4$ (therefore $P(\neg\text{Spam}) = 0.6$)
- $P(\text{CTO}|\text{Spam}) = 0.01$
- $P(\text{CTO}|\neg\text{Spam}) = 0.004$

We can use Bayes theorem to switch conditions:

$$P(\text{Spam}|\text{CTO}) = \frac{P(\text{CTO}|\text{Spam}) \cdot P(\text{Spam})}{P(\text{CTO})}$$

And we can use the Law of Total Probability to find $P(\text{CTO})$:

$$P(\text{CTO}) = P(\text{CTO}|\text{Spam}) \cdot P(\text{Spam}) + P(\text{CTO}|\neg\text{Spam}) \cdot P(\neg\text{Spam})$$

Combining:

$$\begin{aligned} P(\text{Spam}|\text{CTO}) &= \frac{P(\text{CTO}|\text{Spam}) \cdot P(\text{Spam})}{P(\text{CTO}|\text{Spam}) \cdot P(\text{Spam}) + P(\text{CTO}|\neg\text{Spam}) \cdot P(\neg\text{Spam})} \\ &= \frac{0.01 \cdot 0.4}{0.01 \cdot 0.4 + 0.004 \cdot 0.6} \\ &= 0.625 \end{aligned}$$

Question 5

The question gives that $\lambda = 4$ particles per minute ($t = 1$, minute therefore $\theta = 4$).

(a)

$$P(X = 3) = \text{Pois}(k = 3; \theta = 4) = e^{-4}\frac{4^3}{3!} = 0.195$$

(b) Use the relation that:

$$P(X \geq 1) = 1 - P(X = 0)$$

Therefore we can solve:

$$P(X \geq 1) = 1 - e^{-4}\frac{4^0}{0!} = 0.98$$

(c) The new timestep in this question is 2 minutes, therefore $\theta = 8$

$$P(X = 3) = \text{Pois}(k = 3; \theta = 8) = e^{-8}\frac{8^3}{3!} = 0.0286$$

Question 6

(a) This is a coin-tossing question, so let's use the Binomial distribution. The chance of 3 survivors is given by:

$$\text{Binom}(k = 3; 0.4, 8) = \binom{8}{3} 0.4^3 (1 - 0.4)^{8-3} = 0.279$$

(b) The probability of a success in this question is 0.6. 4 doses are administered

(i) We want the probability that any 3 of the 4 doses are successful:

$$\text{Binom}(k = 3; 0.6, 4) = \binom{4}{3} 0.6^3 (1 - 0.6)^{3-4} = 0.3456$$

(ii) Similarly, for no doses being successful:

$$\text{Binom}(k = 0; 0.6, 4) = \binom{4}{0} 0.6^4 (1 - 0.6)^{3-0} = 0.0256$$

(iii) Think about the odds of flipping the same side of a coin 4 times with the probability of that side coming up being 0.4, this would simply be 0.4^4. Therefore the probability that at least *one* dose works is just:

$$1 - 0.4^4 = 0.9744$$

Alternatively you can use the answer from the previous question $(1 - \text{Binom}(k = 0; 0.6, 4))$

(iv) The probability of at most two doses working is given by:

$$P(\text{Max } 2) = P(0) + P(1) + P(2)$$

Which is simply a sum of points on binomial distributions:

$$P(\text{Max } 2) = \text{Binom}(k = 0; 0.6, 4) + \text{Binom}(k = 1; 0.6, 4) + \text{Binom}(k = 2; 0.6, 4)$$

Question 7

The probability of finding a dolphin with a tracker from the random sample is $P(\text{tracked}) = 0.2$ This question is asking the probability that the scientist fails 4 times before finding a tracked dolphin. Therefore use the geometric distribution:

$$\text{Geom}(k = 4; 0.2) = (1 - 0.2)^{4-1} * 0.2 = 0.1024$$

The second part is asking the probability that the scientist needs to select **more** than 6 dolphins before finding one with a tracking device. This is the same as asking the probability of tossing a coin the same way up 6 times with a probability of 0.8 of landing on that side:

$$P(X > 6) = 0.8^6 = 0.262$$

Note: These two questions are subtly different. The first part is asking the probability that the 5th dolphin is tracked if the first 4 are untracked. The second part is asking the probability that the n^{th} dolphin is tracked given that any $n > 6$ dolpins captured first were untracked.

Question 8

We are given that $\lambda = 3$ per minute, and $t = 1$ minutes. Therefore $\theta = 3$

(a)

$$\text{Pois}(k = 0; \theta = 3) = 0.049$$

(b) Rewrite $t = 2$ minutes. Therefore $\theta = 6$. We want $\text{Pois}(k \geq 2; \theta = 6)$:

$$\text{Pois}(k \geq 2; \theta = 6) = 1 - \text{Pois}(k = 1; \theta = 6) - \text{Pois}(k = 0; \theta = 6) = 0.983$$